

# Hydrodynamics of Nuclear Matter in the Chiral Limit

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## Abstract

Using the Poisson bracket method, we construct the hydrodynamics of nuclear matter in the chiral limit, which describes the dynamics of all low-energy degrees of freedom, including the fluid-dynamical and pionic ones. The hydrodynamic equations contain, beside five Euler equations of relativistic fluid dynamics,  $N_f^2 - 1$  second order equations describing propagating pions and  $N_f^2 - 1$  first order equations describing the advection of the vector isospin charges. We present hydrodynamic arguments showing that the pion velocity vanishes at the second order phase transition at  $N_f = 2$ .

Hydrodynamics [1] is the theory describing the low-frequency, long-wavelength dynamics of liquids (or, in an extended sense, of any system). In this regime most degrees of freedom become irrelevant since they relax during the time characteristic of particle collisions; the only ones that survive are either those related to the conservation laws or the phases of the order parameters of broken continuous symmetries. The simplest example is normal fluids, where hydrodynamic variables arise due to the conservation of energy, momentum, and particle number. In superfluid  $\text{He}^4$  and  $\text{He}^3$ , additional hydrodynamic variables emerge from the symmetry breaking by the condensate. Although in all of these systems the physics is quite complicated at the molecular level, at large scales the hydrodynamic equations have simple forms dictated by the symmetries, the pattern of symmetry breaking, and the conservation laws. Such equations typically involve unknown coefficients, which can be computed from the microscopic theory or measured in experiments.

A similar philosophy is shared by the chiral perturbation theory, which describes the long-distance dynamics of QCD with light quarks [2]. At low energies, QCD is a strongly coupled theory where not much can be computed, at least at this moment, in a reliable fashion. However, well below the chiral scale (about 1 GeV), the dynamics is determined by the chiral Lagrangian, which can be written down knowing only the chiral symmetry and the pattern of chiral symmetry breaking of QCD. To the lowest order, pions are governed by the nonlinear sigma model, the only free parameter of which is the pion coupling constant  $f_\pi$ , which can be determined by matching the predictions of theory with experiment.

In nuclear matter in the chiral limit, the low-energy degrees of freedom include both the fluid-dynamical variables (the energy-momentum tensor) and the chiral ones that describe the massless Goldstone modes arising from the breaking of chiral symmetry. All these degrees of freedom are coupled to each other; therefore, a full hydrodynamic treatment must include all these modes. In this respect, the hydrodynamics of nuclear matter is more similar to that of superfluids than of normal fluids. Treatments so far have largely dealt with the fluid dynamical and chiral variables separately, ignoring the interplay between the two [3].

The purpose of this paper is to construct the hydrodynamic theory of nuclear matter in the chiral limit, capable of describing *all* low-energy degrees of freedom of the latter. The primary place where such a theory can be applied is in the theory of heavy-ion collisions, where pion modes can arise by thermal activation or from the spinodal instabilities at the chiral phase transition.

*Degrees of freedom.*—For definiteness, we will be working in  $N_f = 2$  QCD, but most formulas remain valid at any value of  $N_f$ . As the first approximation, we take the quark masses to be zero,  $m_u = m_d = 0$ . The nonzero pion mass, hopefully, can be taken care of by expanding around the chiral limit. We will assume that the temperature and the chemical

potential are low enough so that the chiral symmetry is broken.

The first step toward a hydrodynamic description is to identify the hydrodynamic variables. As mentioned above, the latter are the densities of conserved charges or the phases of order parameters of broken continuous symmetries. In nuclear matter, different vacua arising from chiral symmetry breaking are characterized by the phase  $\Sigma$  of the condensate, where  $\Sigma \in \text{SU}(2)$ , which transforms under chiral rotations as  $\Sigma \rightarrow L\Sigma R^\dagger$ . Slow variations of  $\Sigma$  correspond to soft pions, which, due to the derivative coupling, relax much slower than a typical mode, and, hence, are hydrodynamic degrees of freedom. The conservation laws include those of the energy and the momentum, as well as the baryon number  $n = \int d\mathbf{x} q^\dagger q$  ( $q$  denotes quarks), and the left- and right-handed isospin charges  $\int d\mathbf{x} \rho_{L,R}^a(\mathbf{x}) = \int d\mathbf{x} q_{L,R}^\dagger \frac{\lambda^a}{2} q_{L,R}$ , which are the generators of left and right isospin rotations. Therefore, the hydrodynamics variables are the energy and momentum densities  $T^{0\nu}$ , the baryon number density  $n(\mathbf{x})$ , the left and right isospin charge densities  $\rho_{L,R}^a(\mathbf{x})$ , and the  $\text{SU}(2)$  phase of the condensate  $\Sigma(\mathbf{x})$ .

*Hydrodynamics of a reduced set of variables.*—A nontrivial hydrodynamics can be already constructed for a reduced set of variables, chosen here to contain  $\rho_{L,R}^a$  and  $\Sigma$ . In this simplified treatment, we ignore the variation of the fluid dynamical degrees of freedom, regarding the fluid as frozen. Such a theory is not the full theory, but its construction is much simpler and the equations obtained are suggestive of those in full hydrodynamics, which will be constructed later.

The method we use to write down the hydrodynamic equations is the elegant Poisson bracket technique [4,5], which regards the ideal hydrodynamics of any fluid as a Hamiltonian system defined by a Hamiltonian and a set of Poisson brackets between hydrodynamic variables. The Poisson brackets arise from the canonical commutators in the microscopic theory; the ones between  $\rho_{L,R}$  are the classical version of the current algebra commutators,

$$\begin{aligned} \{\rho_L^a(\mathbf{x}), \rho_L^b(\mathbf{y})\} &= -f^{abc} \rho_L^c(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}), \\ \{\rho_R^a(\mathbf{x}), \rho_R^b(\mathbf{y})\} &= -f^{abc} \rho_R^c(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}), \\ \{\rho_L^a(\mathbf{x}), \rho_R^b(\mathbf{y})\} &= 0 \end{aligned} \tag{1}$$

while those between  $\rho_{L,R}$  with  $\Sigma$  are defined by the transformation laws of  $\Sigma$  under chiral rotations, since  $\rho_{L,R}$  are the densities of charges that generate these transformations,

$$\begin{aligned} \{\rho_L^a(\mathbf{x}), \Sigma(\mathbf{y})\} &= -i \frac{\lambda^a}{2} \Sigma(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}), \\ \{\rho_R^a(\mathbf{x}), \Sigma(\mathbf{y})\} &= i \Sigma(\mathbf{x}) \frac{\lambda^a}{2} \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{2}$$

Finally, the Poisson brackets of  $\Sigma$  with itself vanishes,

$$\{\Sigma(\mathbf{x}), \Sigma(\mathbf{y})\} = 0. \quad (3)$$

Now let us turn to the Hamiltonian. We will limit ourselves to the leading order in derivatives of  $\Sigma$ , since we are interested in the dynamics at the largest scales. We will also keep only leading-order terms in powers of  $\rho$ . The most general form of the Hamiltonian consistent with chiral symmetry is

$$H = \int d\mathbf{x} \left[ \frac{f_s^2}{4} \text{tr} \partial_i \Sigma^\dagger \partial_i \Sigma + \frac{1}{f_t^2} \text{tr} (\rho_L - \Sigma \rho_R \Sigma^\dagger)^2 + \frac{1}{f_v^2} \text{tr} (\rho_L + \Sigma \rho_R \Sigma^\dagger)^2 \right], \quad (4)$$

where  $\rho_{L,R} = \rho_{L,R}^a \frac{\lambda^a}{2}$  and  $f_s$ ,  $f_t$ , and  $f_v$  are constants with the dimension of mass, whose physical meaning will become clear later.

Taking the Poisson brackets with the Hamiltonian (4), we obtain the equations of motion of the hydrodynamic variables,

$$\partial_t \Sigma = -\frac{2i}{f_t^2} (\rho_L \Sigma - \Sigma \rho_R), \quad (5)$$

$$\partial_t \rho_L = -\partial_i J_i^L, \quad (6)$$

$$\partial_t \rho_R = -\partial_i J_i^R, \quad (7)$$

where

$$J_i^L = \frac{i}{4} f_s^2 \Sigma \partial_i \Sigma^\dagger, \quad J_i^R = \frac{i}{4} f_s^2 \Sigma^\dagger \partial_i \Sigma.$$

Equations (6) and (7) reflects the conservation of left- and right-handed flavor charges, where  $J_i^{L,R}$  plays the role of the chiral isospin currents. Note that  $f_v$  does not appear in Eqs. (5)–(7); the reason is that  $\text{tr}(\rho_L + \Sigma \rho_R \Sigma^\dagger)^2 = \text{tr}(\rho_L^2 + \rho_R^2 + 2\rho_L \Sigma \rho_R \Sigma^\dagger)$  is a Casimir operator. Equations (5)–(7) completely determine the dynamics of  $\rho$  and  $\Sigma$ . These differential equations are of first order in time and describe the evolution of  $3(N_f^2 - 1) = 9$  variables.

*The relation to the nonlinear sigma model.*—The hydrodynamic equations (5)–(7) should be contrasted with the field equations of the nonlinear sigma model, which describes the dynamics of pions at zero temperature and chemical potential. The latter is composed of three equations of second order in time derivative, which can be rewritten as six first order equations describing the evolution of  $\Sigma$  and  $\partial_t \Sigma$ . Therefore our hydrodynamics contains at least three extra degrees of freedom that are not presented in the nonlinear sigma model.

To find the relation of the hydrodynamic equations with the field equations of the nonlinear sigma model, one solves Eq. (5) with respect to  $\rho_L$  and  $\rho_R$  and expresses them via  $\partial_t \Sigma$  and a new variable  $\alpha$ ,

$$\rho_L = -\frac{i}{4} f_t^2 \Sigma \partial_t \Sigma^\dagger + \frac{1}{2} \alpha, \quad (8)$$

$$\rho_R = -\frac{i}{4} f_t^2 \Sigma^\dagger \partial_t \Sigma + \frac{1}{2} \Sigma^\dagger \alpha \Sigma. \quad (9)$$

In particular,  $\alpha = \rho_L + \Sigma \rho_R \Sigma^\dagger$ , therefore in the vacuum where  $\Sigma = 1$ ,  $\alpha$  is the density of isovector charge. The equation of motion for  $\alpha$  reads

$$\dot{\alpha} = -\frac{1}{2}[\Sigma \partial_t \Sigma^\dagger, \alpha], \quad (10)$$

while the equation for  $\Sigma$  is now second order in time derivative,

$$i f_t^2 \partial_t (\Sigma \partial_t \Sigma^\dagger) - i f_s^2 \partial_i (\Sigma \partial_i \Sigma^\dagger) + [\Sigma \partial_t \Sigma^\dagger, \alpha] = 0. \quad (11)$$

Equation (10) always allows  $\alpha = 0$  as a solution. In this case, Eq. (11) reduces to the field equation of the nonlinear sigma model with pion velocity equal  $v_\pi = f_s/f_t$ . Therefore  $f_s$  and  $f_t$  play the role of spatial and temporal pion decay constants, respectively [6]. The nonlinear sigma model can be interpreted as the Hamiltonian system (1)–(4) with the *constraint*  $\rho_L + \Sigma \rho_R \Sigma^\dagger = 0$ .

In general, however,  $\alpha$  needs not vanish. Equation (10) implies that, in the dissipationless limit we are considering,  $\alpha$  only *precesses* with time. Since Eq. (10) is first order in time derivative, once dissipation is included into the theory  $\alpha$  will become a true diffusive mode. In any case, when  $\alpha \neq 0$ , our hydrodynamic equations are different from the field equations of the nonlinear sigma model.

*Hydrodynamic pion condensation.*—To understand the physical meaning of the term proportional to  $\alpha$  in Eq. (11), consider the case where  $\Sigma$  makes only small variations around  $\Sigma = 1$ . As noted above,  $\alpha$  is now the isovector charge. Subsequently,  $\alpha$  can be nonzero if *baryons* are included into the theory. For example, if the baryon background contains more neutrons than protons, then  $\alpha^3 = \rho_p - \rho_n$  is nonzero and negative. It is clear from the discussion above that  $\alpha$  is the only baryonic degree of freedom that enter the hydrodynamic theory.

By expanding around  $\Sigma = 1$ , using  $\Sigma = \exp(i f_t^{-1} \lambda^a \pi^a)$ , one finds the following linearized equation for the pion field on an isospin-asymmetric background:

$$\partial_0^2 \pi^a - v_\pi^2 \partial_i \pi^a + f_t^{-2} f^{abc} \partial_0 \pi^b \alpha^c = 0, \quad (12)$$

Such an equation can also be obtained from the mean-field approximation of the chiral perturbation theory, by replacing in the interaction Lagrangian  $-\frac{1}{2} f_\pi^{-2} f^{abc} \pi^a \partial_\mu \pi^b \bar{N} \gamma^\mu \frac{\lambda^c}{2} N$  the isospin baryon density  $\bar{N} \gamma^0 \frac{\lambda^c}{2} N$  by its mean value  $\alpha^c$ . We have shown that this mean-field procedure becomes reliable in the hydrodynamic limit, provided the in-medium pion decay constants  $f_t$  and  $f_s$  are used. Equation (12) predicts a split between the dispersion relations of  $\pi^+$  and  $\pi^-$  in neutron-rich backgrounds where  $\alpha^3 < 0$ . This is the hydrodynamic equivalence of pion condensation [7].

*Pion velocity near the second order phase transition.*—Let us consider  $N_f = 2$  and assume that the baryon chemical potential  $\mu$  is zero or small enough so that the phase transition in temperature is second order [8]. We will argue here that the pion velocity vanishes at the critical temperature. Indeed, as one approaches the critical temperature  $T_c$ , the dependence of the Hamiltonian (4) on the phase of the condensate  $\Sigma$  should become weaker, and, at  $T = T_c$ ,  $H$  should not depend explicitly on  $\Sigma$ . The latter can happen only if at the critical temperature  $f_s = 0$  and  $f_t = f_v$ . Now since  $f_v$  enters the Hamiltonian (4) as  $\alpha^2/f_v^2$ , it is related to the response of nuclear matter in the *isovector* channel. Such a quantity has no reason to vanish at the phase transition. Therefore one can expect  $f_v \neq 0$  at  $T_c$  and, hence,  $f_t$  also does not vanish at the critical temperature, in contrast to  $f_s$ . Therefore the pion velocity  $v_\pi = f_s/f_t$  approaches zero when  $T \rightarrow T_c$ .

*Full hydrodynamics.*—Now let us turn to the discussion of the full hydrodynamic equations, which contains not only chiral variables but also the fluid dynamical ones. These equations can also be derived from the general Poisson bracket technique similar to the one used above. The hydrodynamic variables include, beside  $\Sigma$  and  $\rho_{L,R}$ , the baryon density  $n(\mathbf{x})$ , the entropy density  $s(\mathbf{x})$ , and the momentum density  $T^{0k}(\mathbf{x})$ . The nonvanishing Poisson brackets, beside those written in Eqs. (1)–(3), are

$$\begin{aligned} \{T^{0i}(\mathbf{x}), A(\mathbf{y})\} &= A(\mathbf{x})\partial_i\delta(\mathbf{x}-\mathbf{y}), \quad A = s, n, \rho_{L,R}, \\ \{T^{0i}(\mathbf{x}), \Sigma(\mathbf{y})\} &= -\partial_i\Sigma(\mathbf{x})\delta(\mathbf{x}-\mathbf{y}), \\ \{T^{0i}(\mathbf{x}), T^{0k}(\mathbf{y})\} &= \left[T^{0k}(\mathbf{x})\frac{\partial}{\partial x^i} - T^{0i}(\mathbf{y})\frac{\partial}{\partial y^k}\right]\delta(\mathbf{x}-\mathbf{y}). \end{aligned}$$

In particular, now  $\text{tr}(\rho_L + \Sigma\rho_R\Sigma)^2$  is no longer a Casimir of the Poisson algebra: it has a nonzero Poisson bracket with  $T^{0k}$ . The Hamiltonian is chosen in the most general form consistent with symmetries and containing only second order of  $\rho_{L,R}$  and derivatives of  $\Sigma$ ,

$$\begin{aligned} H = \int d\mathbf{x} T^{00}(\mathbf{x}) = \int d\mathbf{x} \Big( &\epsilon + f_{ij}\text{tr}\partial_i\Sigma\partial_j\Sigma^\dagger + \frac{a}{2}\text{tr}(\rho_L - \Sigma\rho_R\Sigma^\dagger)^2 + \frac{a_v}{2}\text{tr}(\rho_L + \Sigma\rho_R\Sigma^\dagger)^2 \\ &- ic_k\text{tr}(\rho_L\Sigma\partial_k\Sigma^\dagger + \Sigma^\dagger\partial_k\Sigma\rho_R) \Big), \end{aligned} \quad (13)$$

where  $\epsilon$ ,  $f_{ij}$ ,  $a$ ,  $a_v$ , and  $c_k$  are functions of  $s$ ,  $n$  and  $T^{0k}$ . It is, however, more convenient to work with the conjugate variables: the temperature  $T = \partial T^{00}/\partial s$ , the chemical potential  $\mu = \partial T^{00}/\partial n$ , and the velocity  $v^k = \partial T^{00}/\partial T^{0k}$ . In the frame where  $v^k = 0$ ,  $\epsilon$  is determined by the nuclear equation of state, while  $f_{ij} = \delta_{ij}f_s^2/4$ ,  $a = 2f_s^{-2}$ ,  $a_v = 2f_v^{-2}$  are functions of the local temperature  $T$  and chemical potential  $\mu$ , and  $c_k = 0$ . The equations of motion for our variables can be found by taking the Poisson brackets with the Hamiltonian (13). The condition that  $\partial_0 T^{00} = -\partial_i T^{0i}$  allows one to determine the velocity dependence of  $\epsilon$ ,  $f_{ij}$ ,  $a$ ,

$a_v$ , and  $c_k$ , which turns out to be equivalent to the condition of boost invariance. The latter allows the final equations to be written in a relativistically covariant form, although our formalism is Hamiltonian in nature. After quite laborious, but straightforward, calculations [9], one finds that the full set of hydrodynamic equations consists of (i) a second order equation for  $\Sigma$ ,

$$i\partial_\mu[(f_t^2 - f_s^2)u^\mu u^\nu \Sigma \partial_\nu \Sigma^\dagger + f_s^2 \Sigma \partial^\mu \Sigma^\dagger] + [u^\mu \Sigma \partial_\mu \Sigma^\dagger, \alpha] = 0; \quad (14)$$

(ii) a first order equation describing the advection and precession of the isovector charge  $\alpha$ ,

$$\partial_\mu(u^\mu \alpha) = -\frac{1}{2}[u^\mu \Sigma \partial_\mu \Sigma^\dagger, \alpha]; \quad (15)$$

(iii) the continuity equation for the baryon charge,

$$\partial_\mu(u^\mu n) = 0; \quad (16)$$

and (iv) the conservation of energy momentum,

$$\partial_\mu T^{\mu\nu} = 0, \quad (17)$$

where the energy-momentum tensor  $T^{\mu\nu}$  is a sum of a fluid dynamical part and a field (pion) part,

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu - pg^{\mu\nu} + \frac{f_s^2}{4}\text{tr}(\partial^\mu \Sigma \partial^\nu \Sigma^\dagger + \partial^\nu \Sigma \partial^\mu \Sigma^\dagger).$$

Moreover, both the energy density  $\rho$  and the pressure  $p$  receive a contribution from the pion field  $\Sigma$  and the density of isovector charge  $\alpha$ . In particular,

$$p = p_0 + \frac{1}{4}(f_t^2 - f_s^2)u^\mu u^\nu \text{tr}\partial_\mu \Sigma \partial_\nu \Sigma^\dagger + \frac{f_s^2}{4}\text{tr}\partial_\mu \Sigma \partial^\mu \Sigma^\dagger + f_v^{-2}\text{tr}\alpha^2, \quad (18)$$

where  $p_0 = p_0(T, \mu)$  is the pressure at  $\Sigma = 1$  and  $\alpha = 0$ , and  $\rho$  is related to  $p$  by Legendre transformation,

$$\rho = \rho_0 + \frac{1}{4}(\hat{K} + 1)(f_t^2 - f_s^2)u^\mu u^\nu \text{tr}\partial_\mu \Sigma \partial_\nu \Sigma^\dagger + (\hat{K} - 1)\frac{f_s^2}{4}\text{tr}\partial_\mu \Sigma \partial^\mu \Sigma^\dagger + f_v^{-4}(\hat{K} + 1)f_v^2 \text{tr}\alpha^2, \quad (19)$$

where  $\hat{K} = T\partial/\partial T + \mu\partial/\partial\mu$ . As in relativistic fluid dynamics, it can be shown that the conservation of entropy is a consequence of energy-momentum conservation (17), baryon number conservation (16), and thermodynamic relations. The conservation of left- and right-handed isospin currents,

$$\begin{aligned}
J_L^\mu &= -\frac{i}{4}[(f_t^2 - f_s^2)u^\mu u^\nu \Sigma \partial_\nu \Sigma^\dagger + f_s^2 \Sigma \partial^\mu \Sigma^\dagger] + \frac{1}{2}u^\mu \alpha, \\
J_R^\mu &= -\frac{i}{4}[(f_t^2 - f_s^2)u^\mu u^\nu \Sigma^\dagger \partial_\nu \Sigma + f_s^2 \Sigma^\dagger \partial^\mu \Sigma] + \frac{1}{2}u^\mu \Sigma^\dagger \alpha \Sigma,
\end{aligned}$$

follows from Eqs. (14) and (15).

Equations (14) and (15) are the straightforward generalization of Eqs. (10) and (11). At small temperatures and chemical potentials,  $f_t \approx f_s$  and  $\alpha \approx 0$ , the equation of motion for  $\Sigma$  decouples from the fluid flow and coincides with the field equation of the nonlinear sigma model. When  $\Sigma = 1$ , the hydrodynamic equations reduce to those of relativistic fluid dynamics. Thus, our equations include both the relativistic fluid dynamics and the nonlinear sigma model as special cases. This makes our equations ideal for the study of the evolution of disoriented chiral condensates [10] and any other situation where soft pions are in multiply occupied states. For example, the question of coupling between sound waves and pions can be addressed using these equations.

The treatment of this paper could be improved in several directions. First, one can include the effects of quark masses by introducing the mass term  $\text{tr}(M\Sigma) + \text{h.c.}$  into the Hamiltonian, where  $M$  depends on  $T$  and  $\mu$ . Because of the relatively large pion mass, such a term could be quite important. This will modify Eq. (14) for  $\Sigma$  and give additional contributions to the pressure and the energy density in Eqs. (18) and (19), but the main features of the theory remain unchanged [9]. Other next-to-leading terms in the chiral expansion can be also added. Second, the dissipative effects can be introduced, although one will certainly encounter the well-known problems of ambiguity and instability of viscous relativistic fluid dynamics [11].

In conclusion, let us notice that the method developed in this paper should be applicable, with some modifications, to the color-flavor-locking phase of finite-density QCD with  $N_f = 3$  [12]. The only difference is the breaking of  $U(1)$  baryon symmetry. The hydrodynamics contains one additional hydrodynamic variable [the  $U(1)_B$  phase], which gives rise to the superfluid baryon number current.

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